REFLECTION AND TRANSMISSION PROPERTIES OF A STRATIFIED PLASMA*

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ABSTRACT. Equations are derived for the reflection and transmission coefficients of a plane electromagnetic wave incident normally on a plasma having a homogeneous layer in the middle and inhomogeneous layers on both sides. The equations are expressed in terms of parameters which can be calculated from the characteristic properties of the plasma by applying a numerical method and using a digital computer. It is found to be possible to simplify the calculations if the inhomogeneous layers are identical.

A plasma with a trapezoidal distribution of electron density along its thickness is discussed. The reflection and transmission coefficients for a number of such plasmas can be directly calculated from the results reported in the paper. The results refer to a few typical thicknesses of the inhomogeneous layers of the plasma, for which approximate methods that have been generally used in the past for determining the coefficients turn out to be unsatisfactory.

INTRODUCTION

Over the last four or five decades the interaction of an electromagnetic wave and a gaseous plasma has attracted an appreciable amount of attention particularly as due to radio-wave exploration of the ionosphere,—which can be treated consisting of a few layers of plasma,, -in connection with long distance radio-wave communication. In recent years the knowledge of the subject is finding wider application in various fields, such as the earth-to-satellite communication systems, microwave diagnostic methods for plasmas in thermo-nuclear experiments and in discharge and shock-wave tubes (Francis, 1960). The method for determining the reflection and transmission coefficients of a plane electromagnetic wave incident normally on a plasma with an arbitrary distribution of electron density has been described elsewhere (Nicoll and Basu, 1962). However, in practice, the electron density in a plasma can be often assumed to be uniform except in the boundary regions, where the electron density gradually changes from zero to that of the uniform film. Such a plasma can be treated as a stratified one with a uniform film in the middle and inhomogeneous boundary layers on both sides. The reflection and transmission properties of the plasma can be expressed in terms of : (1) the thickness of and propagation coefficient for the

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uniform film, and (2) the reflection and transmission coefficients for the inhomogeneous layers. This has the advantage that the calculation of the reflection and transmission coefficients for plasmas with a central uniform layer of variable thickness but with the same inhomogeneous layers becomes easy. Further simplifications in the calculations are possible if the inhomogeneous layers on either side are identical.

The distribution of electron density in plasmas obtained in discharge tubes is, in many cases, trapezoidal (Wharton and Slager, 1960). This paper reports numerical results in graphical form, which may be used to give the reflection and transmission coefficients for a wide range of plasmas with trapezoidal distribution of electron density. The thicknesses of the inhomogeneous layers of the plasmas are taken to be of the order of a wavelength of the incident wave, since it is in these cases that approximate treatments, such as the Wentzel-Kramers-Brillouin method, become unsatisfactory.

To make the calculations tractable several simplifying assumptions are made. The plasma layers which are of finite thickness are taken to be infinite in lateral directions so that diffraction effects can be neglected. The plasma is assumed to be free from any magnetic field except that of the incident electromagnetic radiation, and the variation of electron density in the plasma is assumed one-dimensional, viz., only along its thickness.

GENERAL EQUATIONS*

In Fig. 1, the section AD represents a stratified medium, consisting of a uniform film BC, marked 2, and inhomogeneous boundary layers AB and CD. The inhomogeneous layers separate the uniform film from the homogeneous media 1 and 3.

*LIST OF PRINCIPAL SYMBOLS

e_e = Charge on an electron.
l = Thickness of a uniform film.
l' = Thickness of an inhomogeneous layer.
m_e = Mass of an electron.
N = Number density of electrons.
N_c = Critical number density of electrons $= \frac{\varepsilon_0 m_e \omega^2}{e_e^2}$ in a rationalized system.
N_u = Number density of electrons in a uniform plasma.
n = Refractive index of a plasma.
n_u = Refractive index of a uniform plasma.
y = Wave admittance normalized with respect to that of free space.
γ_0 = Propagation coefficient in free space.
γ = Propagation coefficient in a plasma.
γ_u = Propagation coefficient in a uniform film.
ε_0 = Absolute permittivity of free space.
λ_0 = Free-space wavelength.
ν = Frequency of collisions of electrons with heavy particles.
ρ = Reflection coefficient for the electric field.
τ = Transmission coefficient for the electric field.
ω = Angular wave-frequency.

Let γ_u be the propagation coefficient in the uniform film and l be its thickness. Let ρ_{12} and τ_{12} represent the reflection and transmission coefficients, for the layer AB, of a plane electromagnetic wave incident normally from the medium 1 under the condition that both the media 1 and 2 are semi-infinite, separated by the layer. Let ρ_{21} and τ_{21} represent the coefficients of a wave incident normally from the medium 2 under the same condition as before. Similarly, for the layer BC, let ρ_{23} and τ_{23} be the coefficients of a wave incident from the medium 2 under the condition that the media 2 and 3 are semi-infinite, separated by BC. If the propagation coefficients at all points in AB and CD are known, the above reflection and transmission coefficients can be calculated by a numerical method described elsewhere in detail (Nicoll and Basu, 1962).

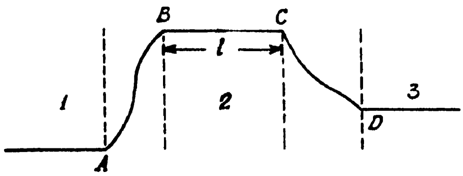


Fig. 1. Stratified medium consisting of a uniform film and inhomogeneous boundary layers.
BC, marked 2—Uniform film.
AB, CD—inhomogeneous boundary layers.
1, 3—homogeneous media.

Now, let a plane electromagnetic wave be incident normally on the whole stratified medium AD from the medium 1. The over-all reflection and transmission coefficients are given by the following equations (see Appendix 1 for the derivation of the equations).

$$\rho = \rho_{12} + \frac{\tau_{12}\tau_{21}\rho_{23}e^{-2\gamma_u l}}{1-\rho_{21}\rho_{23}e^{-2\gamma_u l}} \quad \ldots \quad (1)$$

$$\tau = \frac{\tau_{12}\tau_{23}e^{-\gamma_u l}}{1-\rho_{21}\rho_{23}e^{-2\gamma_u l}} \quad \ldots \quad (2)$$

Let us assume that AD in Fig. 1 represents a stratified plasma. The propagation coefficient in a plasma is given by (Ratcliffe, 1959)

$$\gamma = \gamma_0 n$$

$$= \gamma_0 \left[1 - \frac{N/N_c}{1-i(\nu/\omega)} \right]^{\frac{1}{2}} \quad \ldots \quad (3)^*$$

*For definitions of quantities see the List of Principal Symbols.

Hence, if the electron density and collision frequency at all points in the plasma are known, the propagation coefficients for a wave of angular frequency ω can be calculated from eqn. (3), and the over-all reflection and transmission coefficients can then be determined from eqns. (1) and (2).

INHOMOGENEOUS LAYERS OF ZERO THICKNESS

If the inhomogeneous layers AB and CD in Fig. 1 disappear, leaving the uniform film BC between the media 1 and 3, the quantities ρ_{12}, τ_{12}, ρ_{21}, τ_{21}, ρ_{23} and τ_{23} are all given by the Fresnel formulae. We find that eqns. (1) and (2) are now converted into the well-known formulae for the reflection and transmission coefficients for a uniform film (Born and Wolf, 1959).

$$\rho = \frac{\rho_{12}+\rho_{23}e^{-2\gamma_u l}}{1+\rho_{12}\rho_{23}e^{-2\gamma_u l}} \qquad \cdots \quad (4)$$

$$\tau = \frac{\tau_{12}\,\tau_{23}e^{-\gamma_u l}}{1+\rho_{12}\rho_{23}e^{-2\gamma_u l}} \qquad \cdots \quad (5)$$

In this trivial case the quantities ρ_{12}, τ_{12}, etc. are directly obtained from the refractive indices of the media 1, 2 and 3.

IDENTICAL INHOMOGENEOUS LAYERS

Let us assume that in Fig. 1, the media 1 and 3 are the same and that the inhomogeneous layers AB and DC are identical. This can usually be taken as a valid assumption if AD represents a plasma bounded on either side by free space or air. The over-all reflection and transmission coefficients are, from eqns. (1) and (2),

$$\rho = \rho_{12}+\frac{\tau_{12}\,\tau_{21}\rho_{21}e^{-2\gamma_u l}}{1-(\rho_{21}e^{-\gamma_u l})^2} \qquad \cdots \quad (6)$$

$$\tau = \frac{\tau_{12}\tau_{21}e^{-\gamma_u l}}{1-(\rho_{21}e^{-\gamma_u l})^2} \qquad \cdots \quad (7)$$

Numerical calculations can be simplified by using the relations derived in Appendix 2 and enumerated below.

First we have

$$\tau_{12}/y_1 = \tau_{21}/y_2 \qquad (8)$$

where y_1 and y_2 are the wave admittances of the media 1 and 2, normalized with respect to the wave admittance of free space. Hence, if either of τ_{12} and τ_{21} is known, the other one can be readily determined. It is to be noted here that if the permeability of a medium is unity, the normalized wave admittance is equal to its refractive index. In particular, a plasma is diamagnetic, but only weekly so, and its permeability can be assumed to be unity.

Let us next make a simplifying assumption that the wave admittance is real all throughout the media under consideration. For a plasma the assumption implies that the electron collision frequency is very small, compared to the frequency of the incident wave, and that nowhere in the plasma the electron density exceeds the so-called critical value for the particular wave-frequency. The following relations are now found to hold good.

$$|\rho_{12}| = |\rho_{21}| \qquad \ldots \quad (9)$$

$$\angle\rho_{12} + \angle\rho_{21} = -\pi + 2\angle\tau_{12} \qquad \ldots \quad (10)$$

$$|\tau_{12}|/y_1 = |\tau_{21}|/y_2 \qquad \ldots \quad (11)$$

$$\angle\tau_{12} = \angle\tau_{21} \qquad \ldots \quad (12)$$

If, therefore, numerical calculations are made for one of the pairs, (ρ_{12}, τ_{12}) or (ρ_{21}, τ_{21}), the other pair can be determined from the above equations.

LINEAR DISTRIBUTION OF ELECTRON DENSITY IN INHOMOGENEOUS LAYERS

In many practical cases a plasma can be reasonably assumed to have a trapezoidal distribution of electron density, as shown in Fig. 2 (Wharton and Slager, 1960). Such a plasma can be thought of as having a uniform film in the middle and inhomogeneous layers with linear distribution of electron density on either side.

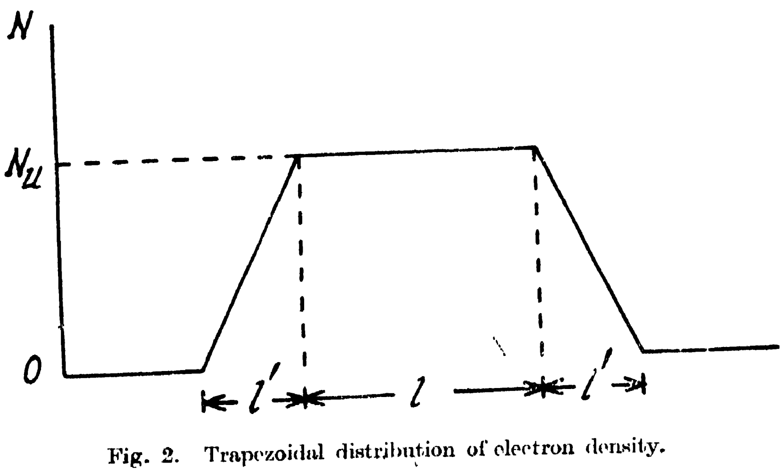


Fig. 2. Trapezoidal distribution of electron density.

Let us consider an inhomogeneous plasma layer $A'B'$ of thickness l' and with linear density distribution, separating free space from a semi-infinite uniform plasma with electron density N_u (Fig. 3). The reflection and transmission coefficients for the layer are given below in graphical form, the thickness l' being taken as $\lambda_0/2$, λ_0 or $2\lambda_0$ where λ_0 represents the free-space wavelength of the incident electromagnetic radiation. The coefficients were obtained by a step-by-step integration process (Nicoll and Basu, 1962) carried out on the Mercury computer at Manchester University. For the sake of simplicity the collision frequency of electrons is assumed to be negligible, compared to the frequency of the incident wave. For plasmas with trapezcidal distribution of electron density and with inhomogeneous layers of thickness $\lambda_0/2$, λ_0 or $2\lambda_0$, the over-all reflection and transmission coefficients can be readily calculated from the given results by using eqns. (1) and (2).

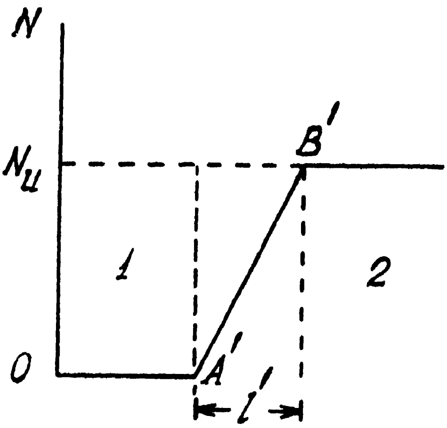


Fig. 3. Inhomogeneous plasma layer with linear distribution of electron density, separating free space from a uniform plasma.
$A'B'$—inhomogeneous plasma layer.
1—free space.
2—uniform pl ma.

It may be mentioned that the results which were calculated on the basis of a numerical method could also be obtained by the algebraic method devised by Hartree (1928-29) for determining the reflection coefficient and extended by the author (Basu, 1960) for determining the transmission coefficient as well.

(a) *Wave incident from free space*

Figures 4(a) and (b) show respectively the magnitude and phase angle of the reflection coefficient, $|\rho_{12}|$ and $\angle\rho_{12}$, for different thicknesses of the inhomogeneous layer $A'B'$ when a plane electromagnetic wave is incident normally on the layer from free space. The magnitude and phase angle of the transmission coefficient, $|\tau_{12}|$ and $\angle\tau_{12}$, are shown in Figs. 4(*c*) and (d). A few salient features of the plots in Figs. 4(a) through (d) are discussed below.

Fig. 4(a) : $|\rho_{12}|$ increases with the increase of N_u until N_u reaches N_c, the critical density. Once the critical density is reached, all the incident wave is reflected, and $|\rho_{12}|$ remains at unity for further increase of N_u. The greater the thickness of the inhomogeneous layer, the smoother the transition from free

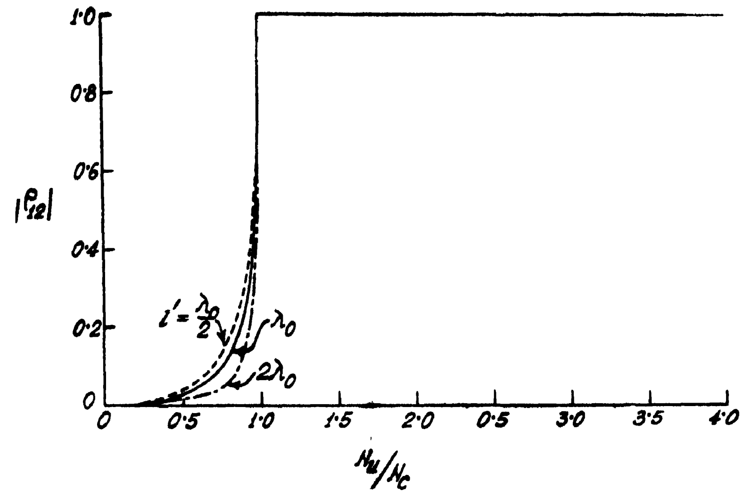
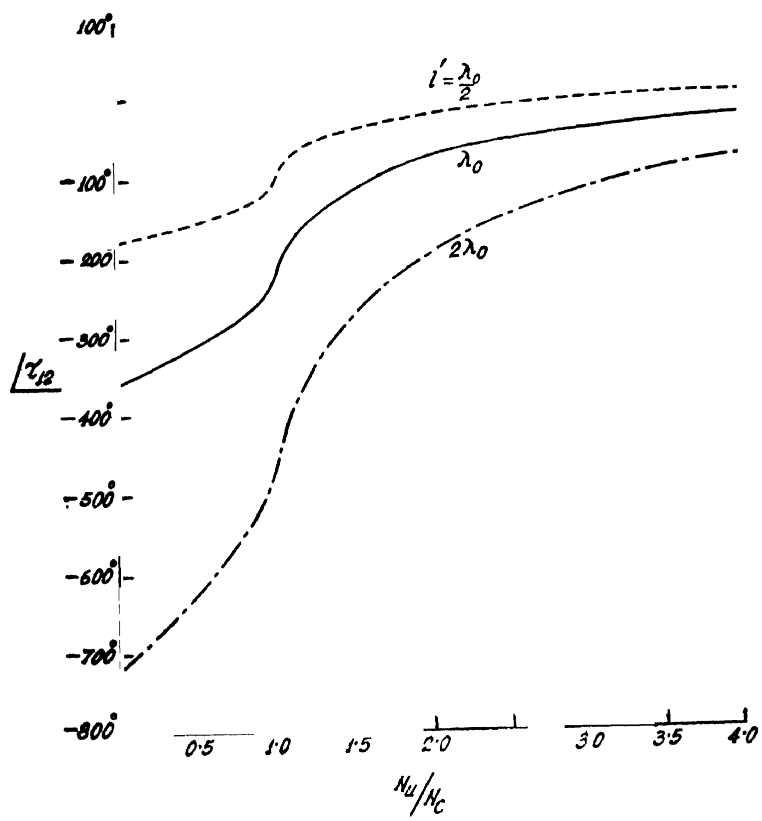


Fig. 4. Characteristics for the layer $A'B'$ in Fig. 3 when a plane wave is incident normally from free space.

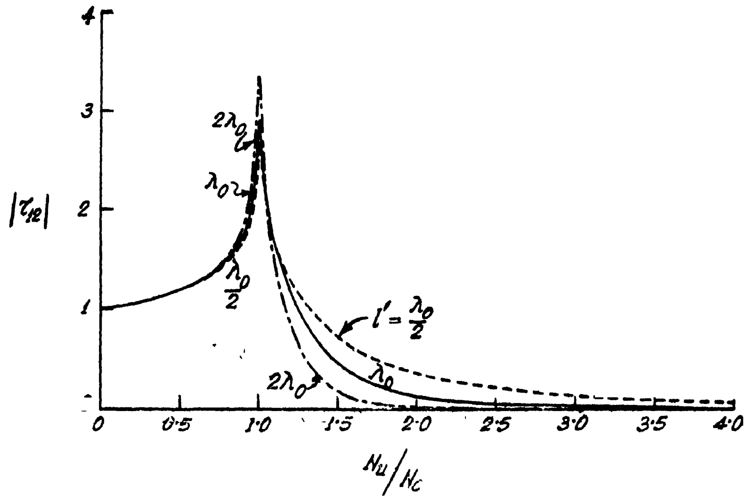
(a) Magnitude of the reflection coefficient.



(b) Phase angle of the reflection coefficient.

space to the uniform plasma and hence the less the the value of $|\rho_{12}|$ for the same electron density N_u when $N_u < N_c$.

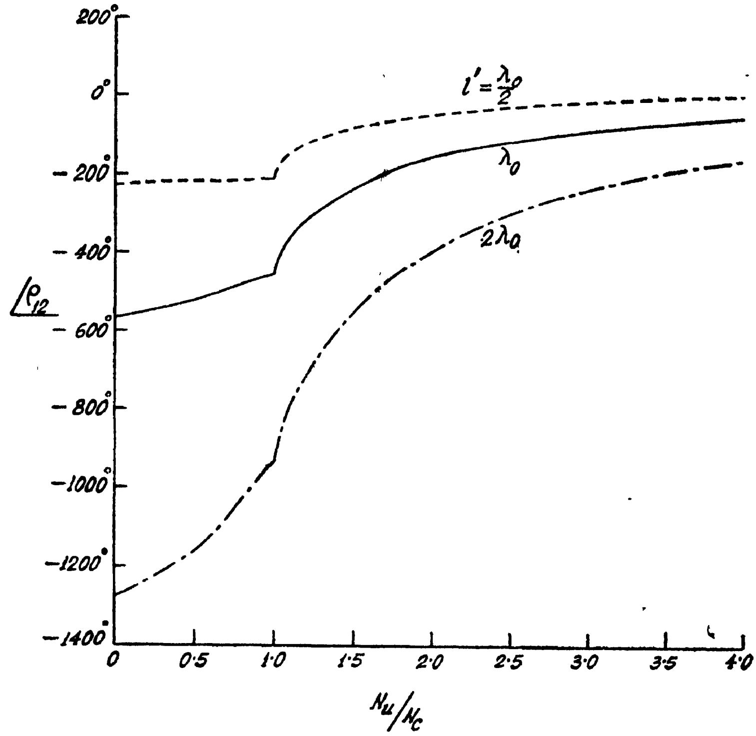
Fig. 4(b) : The reflection takes place, in general, from all parts of the inhomogeneous layer, and the net reflected wave first lags behind the incident wave.



(c) Magnitude of the transmission coefficient.

The lagging angle, i.e., $-\angle\rho_{12}$ decreases as N_u increases. When N_u equals N_c, there is a discontinuity in the curvature of the plot for $\angle\rho_{12}$. This is the point after which the refractive index of a part of the plasma becomes imaginary. When N_u tends toward infinity, the electron density even at the input boundary is very high, and the wave is effectively reflected right at that boundary ; now the reflection angle approaches 180° in the limit (cf. Nicoll and Basu, 1962). If N_u is kept fixed and the thickness l' is increased, $-\angle\rho_{12}$ increases. However, the difference between the angles is reduced as N_u rises, the difference being zero when $N_u = \infty$.

Fig. 4(c) : When $N_u < N_c$, the magnitude of the transmitted electric vector is given by $|\tau_{12}| = (p_t/n_u)^{\frac{1}{2}}$, where p_t is the transmitted power normalized with respect to the incident power and n_u is the refractive index of the uniform plasma. With the increase of N_u from zero, p_t which is equal to $1-|\rho_{12}|^2$ is reduced, but n_u is reduced at a faster rate; as a result $|\tau_{12}|$ increases. When, however, N_u exceeds N_c, the refractive index becomes imaginary in a part of the inhomogeneous layer, and the wave is attenuated; now with the increase of N_u the attenuation increases and $|\tau_{12}|$ is reduced more and more. It is interesting to note that although the transmitted power p_t is zero in this region, the transmitted electric vector is not so; the vector is, in this case, associated with an evanescent wave in the uniform plasma. Let us next consider different thicknesses of the inhomogeneous layer, and let $N_u < N_c$. An increase in the thickness results in a decrease in $|\rho_{12}|^2$ and, therefore, in an increase in p_t. Hence for the same electron density N_u, i.e., for the same refractive index n_u, the greater the thickness l', the higher the value of $|\tau_{12}|$. When $N_u > N_c$, the region over which the wave is attenuated increases with an increase in l', and $|\tau_{12}|$ decreases at a more rapid rate.



(d) Phase angle of the transmission coefficient.

Fig. 4(d) : Initially at $N_u = 0$ the phase angle of the transmitted electric vector, $\angle\tau_{12}$, corresponds to the thickness of the inhomogeneous layer, this angle being negative as the transmitted vector lags behind the incident one. With the increase in N_u the electron density throughout the layer increases, and the refractive index is reduced. This results in a decrease in the effective path-length in the layer, and $-\angle\tau_{12}$ is diminished. At $N_u = N_c$ there is a noticeable change in the curvature of the plot of the phase angle. The greater the thickness l', obviously the higher the initial value of $-\angle\tau_{12}$. The difference between the transmission angles, however, decreases with increase in N_u. All the plots approach 90° as N_u tends to infinity. This is due to the fact that the phase change occurs now mainly at the input boundary, where the limiting value is 90°.

(b) Wave incident from the uniform plasma

Referring to the model in Fig. 3, let us assume that a plane electromagnetic wave is incident normally on the inhomogeneous layer from the semi-infinite uniform plasma. The magnitude and phase angle of the reflection coefficient,

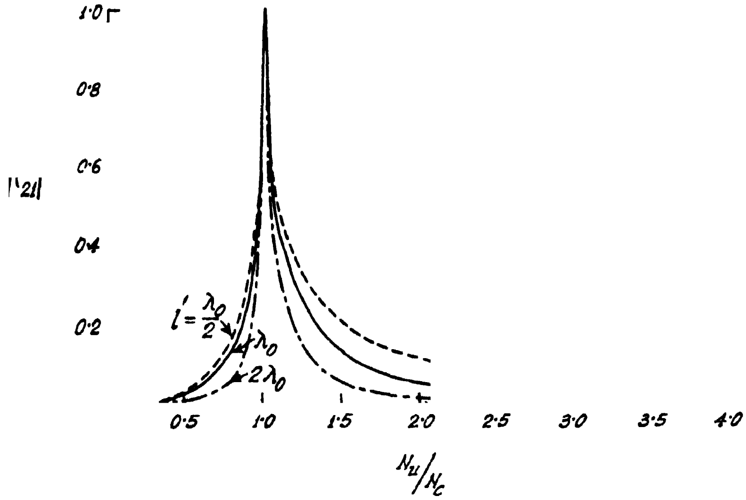
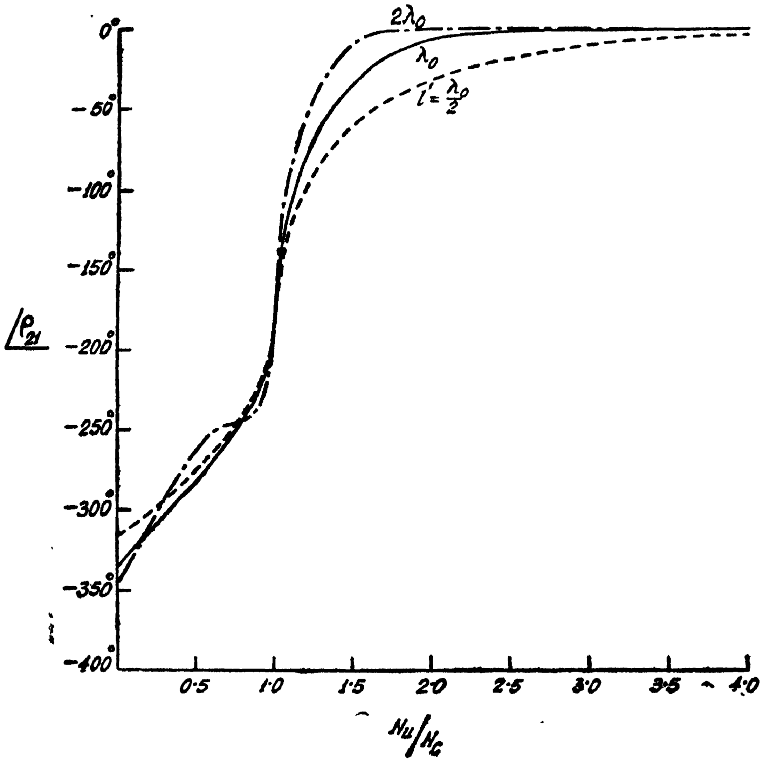


Fig. 5. (a) Characteristics for the layer $A'B'$ in Fig. 3 when a plane wave is incident normally from the uniform plasma.



(b)—represent the same parameters as described below Fig. 4.

$|\rho_{21}|$ and $\angle\rho_{21}$, are plotted in Figs. 5(a) and (b). The corresponding quantities for the transmission coefficient, $|\tau_{21}|$ and $\angle\tau_{21}$, are shown in Figs. 5(c) and (*d*).

The plots in Figs. 4 and 5 conform to the relation given in eqn. (8) in the general case and to those in eqns. (9) through (12) for $N_u < N_c$. The following additional points are worth noting in connection with the plots in Figs. 5(*a*) through (*d*).

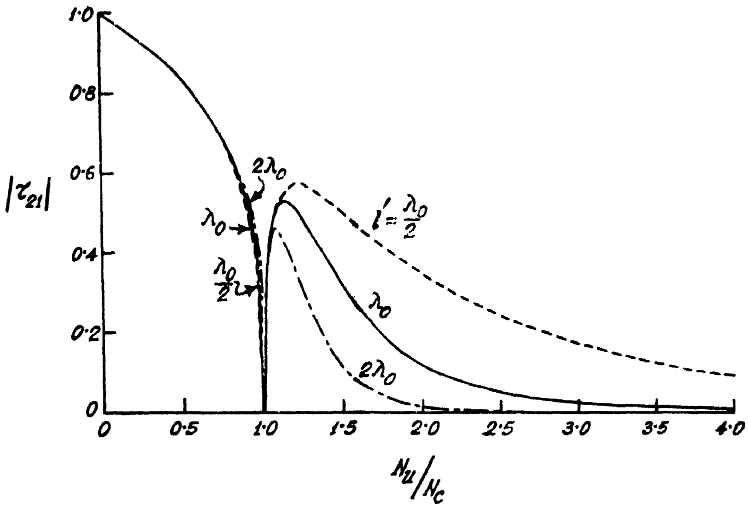


Fig. 5(c)

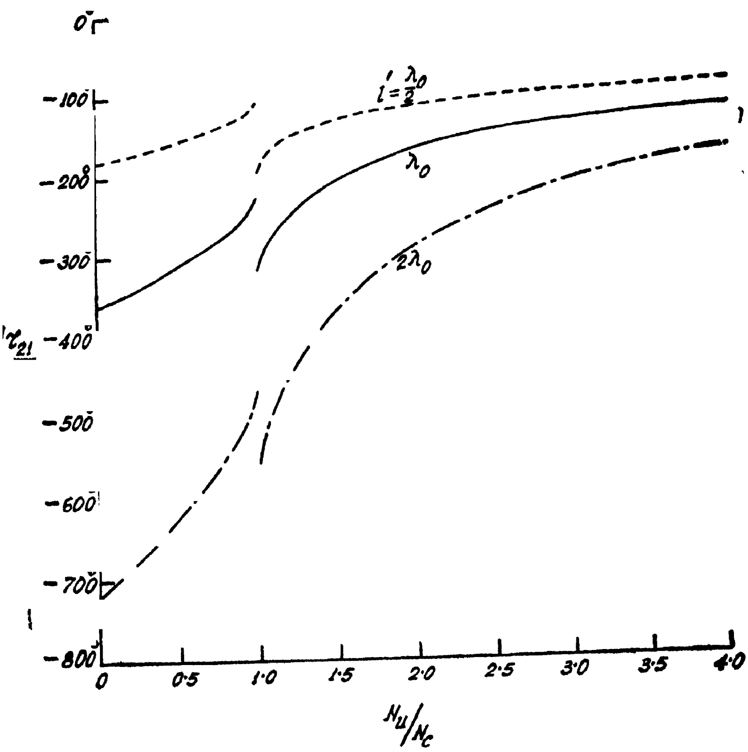


Fig. 5 (d)

(1) Consider the case : $N_u = N_c$.

$\rho_{21} = |1| \angle -180° = -1$, and $\tau_{21} = 0$. The uniform plasma is now perfectly short-circuited, as it were, at the input boundary of the inhomogeneous layer. Since the refractive index of the plasma is zero, there is no change of the electric field in the plasma, and the resultant electric field all throughout is zero. There is an abrupt change of 90° in each plot of $\angle \tau_{21}$, corresponding to the fact that the refractive index of the uniform plasma is real when $N_u < N_c$ and imaginary when $N_u > N_c$.

(2) Consider the case : $N_u > N_c$.

The uniform plasma is reactive in nature, and the energy associated with electromagnetic waves inside the plasma is in the form of stored energy· However, τ_{21} has a finite magnitude, indicating that it is still possible to transmit some power out. This can be explained by the presence of a 'reflected wave' in the plasma. A similar situation arises when a microwave piston attenuator is terminated by a resistive load (Barlow and Cullen, 1950).

If the thickness of the inhomogeneous layer were zero, i.e., if there were a sudden transition from the uniform plasma to free space, $|\rho_{21}|$ would have remained constant at unity as N_u increased beyond N_c, and $|\tau_{21}|$ would have simultaneously increased, reaching 2 at $N_u = \infty$. For a finite thickness of the inhomogeneous layer the net reflected wave can be thought of as the resultant of the waves reflected from all parts of the layer, and on account of the reactive attenuation of the waves in the layer, $|\rho_{21}|$ is reduced with the increase of N_u, as shown in Fig. 5(a); $|\tau_{21}|$ first increases as N_u increases, but as the attenuation becomes predominant, $|\tau_{21}|$ gets reduced with further increase in N_u (Fig. 5c).

As N_u increases beyond N_c, the penetration becomes less and $-\angle \rho_{21}$ decreases, the penetration tending to zero in the limit and $-\angle \rho_{21}$ tending to 0° (Fig. 5b), $-\angle \tau_{21}$ also approaches 0° at the same time (Fig. 5d), as there is, in the limit, no phase change in the whole plasma medium nor any at the output boundary.

(c) *Triangular distribution of the electron density*

If the thickness of the uniform film in Fig. 2 is made equal to zero, the trapezoidal distribution of electron density becomes a triangular one. The overall reflection and transmission coefficients for a few models of the plasma with triangular density distribution have been previously reported by Nicoll and Basu (1962). The results obtained on the basis of eqns. (6) and (7) and the plots in Figs. 4 and 5 are found to be in conformity with the previous results.

CONCLUSIONS

The reflection and transmission coefficients for the whole column AD in Fig. 1 can be obtained by the numerical method described by Nicoll and Basu (1962). However, if the inhomogeneous layers AB and CD remain the same and only the thickness of the uniform film BC changes, it is not necessary to go through the numerical process every time; eqns. (1) and (2) can be, instead, utilized. The reflection and transmission coefficients for some models of plasmas with trapezoidal distribution of electron density can be directly calculated from the results reported in this paper; the inhomogeneous layers of the plasmas must, of course, correspond to those that have been discussed.

The method described in deriving eqns. (1) and (2) can be readily extended to yield the reflection and transmission properties of a system consisting of a number of homogeneous and inhomogeneous layers, e.g., a plasma with trapezoidal distribution of electron density, enclosed by the glass walls of the container.

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APPENDIX I

DERIVATION OF THE GENERAL EQUATONS

Considering Fig. 1, let a plane electromagnetic wave be incident normally on the stratified medium AD from the medium 1. At the points A, B and C there are now two resultant waves, one forward-moving and the other backward-moving. However, at D there is only the forward-moving wave in the medium 3, which corresponds to the wave transmitted out. We assume that at the point D the electric field corresponding to this wave is given by a vector of unit magnitude and zero phase. The reflection and transmission coefficients for the inhomogeneous layers AB and CD are given by ρ_{12}, τ_{12}, etc., as described in Sec. II.

The electric fields of the forward and backward waves in the medium 2 at the point C are :

$$E_{f(c)} = \frac{1}{\tau_{23}} \qquad \ldots \quad (A.1)$$

$$E_{b(c)} = \frac{\rho_{23}}{\tau_{23}} \qquad \ldots \quad (A.2)$$

The corresponding quantities in the same medium at the point B are :

$$E_{f(B)} = \frac{e^{\gamma_u l}}{\tau_{23}} \qquad \ldots \quad (A.3)$$

$$E_{b(B)} = \frac{\rho_{23} e^{-\gamma_u l}}{\tau_{23}} \qquad \ldots \quad (A.4)$$

The forward wave at B can be divided into two parts, the electric field of the first part being $\rho_{21} E_{b(B)}$ and that of the other being $E_{f(B)} - \rho_{21} E_{b(B)}$. The backward wave at B together with the first part of the forward wave corresponds to a backward wave in the medium 1; the electric field of this backward wave at the point A is $\tau_{21} E_{b(B)}$. The second part of the forward wave at B corresponds to a forward wave plus a backward wave in the medium 1; the electric fields of these waves at A are $(\tau_{12})^{-1} (E_{f(B)} - \rho_{21} E_{b(B)})$ and $\rho_{12}(\tau_{12})^{-1}(E_{f(B)} - \rho_{21} E_{b(B)})$ respectively.

Hence the electric fields of the total forward and backward waves in the medium 1 at the point A are :

$$E_{f(A)} = \frac{1}{\tau_{12}} (E_{f(B)} - \rho_{21} E_{b(B)})$$

$$= \frac{e^{\gamma_u l}}{\tau_{12}\tau_{23}} (1 - \rho_{21}\rho_{23} e^{-2\gamma_u l}) \qquad \ldots \quad (A.5)$$

$$E_{b(A)} = \tau_{21} E_{b(B)} + \frac{\rho_{12}}{\tau_{12}} (E_{f(B)} - \rho_{21} E_{b(B)})$$

$$= \frac{\tau_{21}\rho_{23} e^{-\gamma_u l}}{\tau_{23}} + \frac{\rho_{12} e^{\gamma_u l}}{\tau_{12}\tau_{23}} (1 - \rho_{21}\rho_{23} e^{-2\gamma_u l}) \qquad \ldots \quad (A.6)$$

The over-all reflection coefficient for the stratified medium AD is given by

$$\rho = \frac{E_{b(A)}}{E_{f(A)}} \qquad \ldots \quad (A.7)$$

Since the transmitted electric vector at D has been assumed to be of unit magnitude and zero phase, the over-all transmission coefficient is given by

$$\tau = \frac{1}{E_{f(A)}} \qquad \ldots \quad (A.8)$$

The general equations (1) and (2) in Sec. II are derived from eqns. (A.5) through (A.8).

It should be noted that the equations can also be derived in two other ways, as mentioned by Montgomery (1947) in a different context.

APPENDIX 2

RELATIONS BETWEEN THE REFLECTION AND TRANSMISSION COEFFICIENTS FOR AN INHOMOGENEOUS LAYER

Let an inhomogeneous layer lie between two semi-infinite homogeneous media 1 and 2. Here we determine the relations between ρ_{12}, ρ_{21}, τ_{12} and τ_{21} where ρ_{12} and τ_{12} represent the reflection and transmission coefficients, for the inhomogeneous layer, of a plane electromagnetic wave incident normally from the medium 1 while ρ_{21} and τ_{21} represent the corresponding quantities if the wave is incident from the medium 2.

The inhomogeneous layer can be considered as a stratified medium consisting of a very large number of thin films. If the thicknesses of the films are sufficiently small, it is permissible to regard the refractive index to be constant throughout each film.

Now, Born and Wolf (1959) have derived expressions for the reflection and transmission coefficients for a stratified medium in terms of a matrix which is a characteristic property of the particular medium. Let M represent the characteristic matrix for the inhomogeneous layer under consideration. We can relate the electric and magnetic fields at the input and output boundaries, E_i, H_i, and E_0, H_0, in the following way.

$$\begin{bmatrix} E_i \\ H_i \end{bmatrix} = M \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} \qquad \ldots \text{(A·9)}$$

The reflection and transmission coefficients for the inhomogeneous layer, ρ_i and τ_i, can be expressed in terms of the elements of the matrix M.

$$\rho_i = \frac{(m_{11}y_i - m_{22}y_0) + (m_{12}y_iy_0 - m_{21})}{(m_{11}y_i + m_{22}y_0) + (m_{12}y_iy_0 + m_{21})} \qquad \ldots \text{(A·10)}$$

$$\tau_i = \frac{2y_i}{(m_{11}y_i + m_{22}y_0) + (m_{12}y_iy_0 + m_{21})} \qquad \ldots \text{(A.11)}$$

In the above equations y_i and y_0 are the wave admittances of the homogeneous media on the input and output sides of the inhomogeneous layer, normalized with respect to the wave admittance of free space.

Let us consider the thin films constituting the inhomogeneous layer. Let these films, starting from the input boundary, be designated as (1), (2), ... (n), and let $M^{(1)}$, $M^{(2)}$ ... $M^{(n)}$ be their characteristic matrices. It can be shown that

$$M = M^{(1)}M^{(2)} \dots M^{(n)} \qquad \dots(A.12)$$

Let

$$M^{(1)} = \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} m_{11}^{(2)} & m_{12}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} \end{bmatrix}, \text{ etc.}$$

Then, from eqn. (A.12),

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} \end{bmatrix} \begin{bmatrix} m_{11}^{(2)} & m_{12}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} \end{bmatrix} \dots \begin{bmatrix} m_{11}^{(n)} & m_{12}^{(n)} \\ m_{21}^{(n)} & m_{22}^{(n)} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma\, m_{1a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}1}^{(n)} & \Sigma\, m_{1a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}2}^{(n)} \\ \Sigma\, m_{2a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}1}^{(n)} & \Sigma\, m_{2a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}2}^{(n)} \end{bmatrix} \qquad \dots (A.13)$$

where each of a_1, a_2, a_3, ...a_{n-2}, a_{n-1} can be either 1 or 2.

From eqn. (A.13)

$$\left.\begin{aligned} m_{11} &= \Sigma\, m_{1a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}1}^{(n)} \\ m_{12} &= \Sigma\, m_{1a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}2}^{(n)} \\ m_{21} &= \Sigma\, m_{2a_1}^{(1)} m_{a_1a_1}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}1}^{(n)} \\ m_{22} &= \Sigma\, m_{2a_1}^{(1)} m_{a_1a_2}^{(2)} m_{a_2a_3}^{(3)} \dots m_{a_{n-2}a_{n-1}}^{(n-1)} m_{a_{n-1}2}^{(n)} \end{aligned}\right\} \qquad \dots (A.14)$$

Let us first imagine that an electromagnetic wave is incident on the inhomogeneous layer from the medium 1. Let the films extending from the medium 1 to 2 be designated as (1), (2)... (n). Then the characteristic matrix of the whole layer is given by eqn. (A.12), and its elements are given by eqn. (A.14). The reflection coefficient ρ_{12} and the transmission coefficient τ_{12} are obtained from eqns. (A.10) and (A.11).

$$\rho_{12} = \frac{(m_{11}y_1 - m_{22}y_2) - (m_{12}y_1y_2 - m_{21})}{(m_{11}y_1 + m_{22}y_2) + (m_{12}y_1y_2 + m_{21})} \qquad \ldots \text{(A.15)}$$

$$\tau_{12} = \frac{2y_1}{(m_{11}y_1 + m_{22}y_2) + (m_{12}y_1y_2 + m_{21})} \qquad \ldots \text{(A.16)}$$

where y_1 and y_2 are the normalized wave admittances of the media 1 and 2.

Let us next imagine that an electromagnetic wave is incident on the inhomogeneous layer from the medium 2. Starting from the input boundary, the thin films are now (n), $(n-1)$, ...(1). Let M' be the characteristic matrix for the whole layer in this case. Then,

$$M' = M^{(n)} M^{(n-1)} \ldots M^{(1)} \qquad \ldots \text{(A.17)}$$

If

$$M' = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \qquad \ldots \text{(A.18)}$$

it can be easily shown that

$$\left.\begin{aligned} m'_{11} &= \Sigma\, m_{1a_1}{}^{(n)} m_{a_1a_2}{}^{(n-1)} m_{a_2a_3}{}^{(n-2)} \ldots m_{a_{n-2}a_{n-1}}{}^{(2)} m_{a_{n-1}1}{}^{(1)} \\ m'_{12} &= \Sigma\, m_{1a_1}{}^{(n)} m_{a_1a_2}{}^{(n-1)} m_{a_2a_3}{}^{(n-2)} \ldots m_{a_{n-2}a_{n-1}}{}^{(2)} m_{a_{n-1}2}{}^{(1)} \\ m'_{21} &= \Sigma\, m_{2a_1}{}^{(n)} m_{a_1a_2}{}^{(n-1)} m_{a_2a_3}{}^{(n-2)} \ldots m_{a_{n-2}a_{n-1}}{}^{(2)} m_{a_{n-1}1}{}^{(1)} \\ m'_{22} &= \Sigma\, m_{2a_1}{}^{(n)} m_{a_1a_2}{}^{(n-1)} m_{a_2a_3}{}^{(n-2)} \ldots m_{a_{n-2}a_{n-1}}{}^{(2)} m_{a_{n-1}2}{}^{(1)} \end{aligned}\right\} \qquad \ldots \text{(A.19)}$$

where, as in eqn. (A.14), each of $a_1, a_2, a_3, \ldots a_{n-2}, a_{n-1}$ can be either 1 or 2.

Now, following Born and Wolf (1959), the characteristic matrix of the thin film (r) can be written

$$M^{(r)} = \begin{bmatrix} m_{11}{}^{(r)} & m_{12}{}^{(r)} \\ m_{21}{}^{(r)} & m_{22}{}^{(r)} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\gamma_0 n_r \delta l_r) & -\left(\dfrac{i}{y_r}\right)\sin(\gamma_0 n_r \delta l_r) \\ -iy_r \sin(\gamma_0 n_r \delta l_r) & \cos(\gamma_0 n_r \delta l_r) \end{bmatrix} \qquad \ldots \text{(A.20)}$$

where n_r and y_r are respectively the refractive index and normalized wave admittance of the film, δl_r is its thickness, and γ_0 is the propagation coefficient of the incident wave in free space.

It is found from eqn. (A.20) that

$$m_{11}{}^{(r)} = m_{22}{}^{(r)} \qquad \ldots \text{(A.21)}$$

Since eqn. (A.21) is valid for all the thin films, i.e., for all values of (r) from (1) to (n), it can be shown from eqns. (A.14) and (A.19) that

$$\left.\begin{aligned} m'_{11} &= m_{22} \\ m'_{12} &= m_{12} \\ m'_{21} &= m_{21} \\ m'_{22} &= m_{11} \end{aligned}\right\} \qquad \ldots \text{(A.22)}$$

Considering eqns. (A.10), (A.11) and (A.22), the reflection and transmission coefficients, ρ_{21} and τ_{21}, are given by

$$\rho_{21} = \frac{(m'_{11}y_2 - m'_{22}y_1) + (m'_{12}y_2y_1 - m'_{21})}{(m'_{11}y_2 + m'_{22}y_1) + (m'_{12}y_2y_1 + m'_{21})}$$

$$= \frac{-(m_{11}y_1 - m_{22}y_2) + (m_{12}y_1y_2 - m_{21})}{(m_{11}y_1 + m_{22}y_2) + (m_{12}y_1y_2 + m_{21})} \quad \ldots \quad (A.23)$$

$$\tau_{21} = \frac{2y_2}{(m'_{11}y_1 + m'_{22}y_2) + (m'_{12}y_1y_2 + m'_{21})}$$

$$= \frac{2y_2}{(m_{11}y_1 + m_{22}y_2) + (m_{12}y_1y_2 + m_{21})} \quad \ldots \quad (A.24)$$

The relation given in eqn. (8) in Sec. IV is obtained from eqns. (A.16) and (A.24).

If the wave admittance is assumed to be real all throughout the media under consideration, y_1 and y_2 are real. m_{11} and m_{22} can also be shown to be real while m_{12} and m_{21} turn out to be imaginary. Under these conditions eqns. (A.15), (A.16), (A.23) and (A.24) give the relations expressed in eqns. (9), (11) and (12) in Sec. IV. It is also found that

$$\angle\rho_{12} + \angle\rho_{21} = k\pi + 2\angle\tau_{12} \qquad (A.25)$$

where k can be any integer. Since eqn. (A.15) is a general equation, it must be valid for the trivial case when the thickness of the inhomogeneous layer becomes zero; it is then found from Fresnel's formulae that $k = -1$. Hence we get the relation in eqn. (10).

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